

Direct post-Newtonian orbital effects around an oblate body arbitrarily oriented in space

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Abstract

We analytically calculate the direct post-Newtonian orbital perturbations of a test particle moving about an oblate primary endowed with an arbitrarily oriented quadrupole mass moment Q_2 . The resulting long-term rates of change of order $\mathcal{O}(Q_2 c^{-2})$ have a general validity since no a-priori assumptions on both the spin axis $\hat{\mathbf{k}}$ of the central body and the orbital configuration of the orbiter are made. We discuss the indirect, mixed orbital effects of order $\mathcal{O}(Q_2 c^{-2})$ arising from the interplay between the standard Newtonian quadrupolar and post-Newtonian point particle accelerations, and the issue of their potential measurability in actual data reductions.

Keywords:

Experimental studies of gravity;

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1 Introduction

General relativity predicts that the oblateness of a central body of mass M , equatorial radius R and quadrupole mass moment Q_2 has an impact on the orbital motion of a test particle also at the first post-Newtonian (1PN) level [1–6], in addition to the well known purely Newtonian effects [7, 8]. The possibility of actually measuring such PN quadrupolar orbital perturbations with the Juno spacecraft, currently en route to Jupiter, has recently been pointed out [9]. They could also play a role in the long-term evolution of

stellar systems orbiting the supermassive black hole at the center of the Galaxy [6]; the same conclusions arise also from the inclusion of the 1PN corrections in the statistical mechanics of large ensembles [10–12].

The calculations existing in the literature [1–3, 5] refer to the case in which the spin axis of the primary is aligned with the reference z axis of the coordinate system adopted. In view of the fact that in several potentially interesting scenarios the orientation of spinning bodies such as main sequence stars orbited by exoplanets, binaries hosting compact objects, etc., is actually poor known, in this paper we perform a complete calculation of the 1PN quadrupole orbital perturbations without aligning the symmetry axis of the oblate primary with any specific direction.

When the full 1PN equations of motion are considered, 1PN quadrupole orbital effects generally arise not only from a new acceleration proportional to $Q_2 c^{-2}$, where c is the speed of light in vacuum, but also from the mutual interaction of the standard Newtonian quadrupolar acceleration and the 1PN point particle one, causing indirect or mixed effects of order $\mathcal{O}(Q_2 c^{-2})$ [4–6]. In principle, they are of the same order of magnitude of the direct ones. Thus, we investigate if they can have a role from an observational point of view.

The paper is organized as follows. In Section 2, the 1PN equations of motion of a test particle are obtained to the first order in the primary’s quadrupole moment, assumed arbitrarily oriented in space. The direct orbital perturbations are analytically worked out with the Gauss perturbative equations in Section 3. A numerical confirmation of them is obtained by numerically integrating the equations of motion in Section 3.1. The actual measurability of the indirect, mixed orbital perturbations due to the interplay between the 1PN point particle and the Newtonian quadrupolar accelerations in realistic data reductions is discussed in Section 4. Section 5 summarizes our findings.

2 The post-Newtonian quadrupole acceleration of order $\mathcal{O}(Q_2 c^{-2})$

The PN Lagrangian, correct up to second-order terms in c^{-1} is [2]

$$\mathcal{L}_{1\text{PN}} = \frac{1}{2}v^2 - \frac{1}{2}c^2 h_{00} + \frac{v^4}{8c^2} - \frac{1}{4}h_{00}v^2 + \frac{1}{8}c^2 h_{00}^2 - \frac{1}{2}h_{ij}v^i v^j - ch_{0i}v^i + \dots, \quad (1)$$

v is the velocity of the test particle and $h_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$ are the dimensionless corrections to the Galilean spacetime metric coefficients; in the

second term of eq. (1) h_{00} has to be evaluated by keeping the $\mathcal{O}(c^{-4})$ term as well.

By neglecting the gravitomagnetic terms proportional to $h_{0i}, i = 1, 2, 3$, one has [2]

$$h_{00} = -\frac{2U}{c^2} + \frac{2U^2}{c^4}, \quad (2)$$

$$h_{ij} = -\frac{2U}{c^2}\delta_{ij}, \quad i, j = 1, 2, 3, \quad (3)$$

where U is the Newtonian gravitational potential of the source. By inserting eq. (2)-eq. (3) in eq. (1), the following $\mathcal{O}(c^{-2})$ Lagrangian is obtained [4,13]

$$\mathcal{L}_{\text{1PN}} = \frac{1}{2}v^2 + U + \frac{1}{c^2} \left(\frac{v^4}{8} - \frac{U^2}{2} + \frac{3}{2}Uv^2 \right). \quad (4)$$

In the case of an isolated oblate body, its external potential at distance r from its center of mass is

$$U(\mathbf{r}) = U_0(r) + U_{Q_2}(\mathbf{r}), \quad (5)$$

with

$$U_0(r) = \frac{GM}{r}, \quad (6)$$

$$U_{Q_2}(\mathbf{r}) = \frac{GQ_2}{r^3} \left[\frac{3(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}})^2 - 1}{2} \right]. \quad (7)$$

In eq. (6)-eq. (7), G is the Newtonian constant of gravitation, $\hat{\mathbf{k}}$ is the unit vector of the body's spin axis, and $\hat{\mathbf{r}}$ is the unit position vector directed from its center of mass to the external point of interest. The quadrupole Q_2 in eq. (7) has dimensions $[Q_2] = \text{ML}^2$; it is connected with the usual dimensionless even zonal harmonic J_2 of standard geodesy/astronomy textbooks by

$$Q_2 = -J_2MR^2. \quad (8)$$

The test particle acceleration can be obtained from eq. (4) by means of the usual Lagrange equations of motion

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}_{\text{1PN}}}{\partial \mathbf{v}} \right) - \frac{\partial \mathcal{L}_{\text{1PN}}}{\partial \mathbf{r}} = 0. \quad (9)$$

As a result, terms of order $\mathcal{O}(c^{-2})$ containing the second derivatives with respect to time of the spatial coordinates are generated by $d(\partial\mathcal{L}_{\text{1PN}}/\partial\mathbf{v})/dt$. As such, the components of the test particle acceleration, in Cartesian coordinates, are

$$\begin{aligned}\ddot{x} = & -\frac{\ddot{x}v_x^2}{c^2} - \frac{\ddot{y}v_xv_y}{c^2} - \frac{\ddot{z}v_xv_z}{c^2} - \frac{\ddot{x}v^2}{2c^2} - \\ & -\frac{3\ddot{x}GM}{c^2r} - \frac{3\ddot{x}GQ_2}{2c^2r^3} \left[-1 + 3 \left(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}} \right)^2 \right] + \frac{\partial\mathcal{L}_{\text{1PN}}}{\partial x},\end{aligned}\quad (10)$$

$$\begin{aligned}\ddot{y} = & -\frac{\ddot{x}v_xv_y}{c^2} - \frac{\ddot{y}v_y^2}{c^2} - \frac{\ddot{z}v_yv_z}{c^2} - \frac{\ddot{y}v^2}{2c^2} - \\ & -\frac{3\ddot{y}GM}{c^2r} - \frac{3\ddot{y}GQ_2}{2c^2r^3} \left[-1 + 3 \left(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}} \right)^2 \right] + \frac{\partial\mathcal{L}_{\text{1PN}}}{\partial y},\end{aligned}\quad (11)$$

$$\begin{aligned}\ddot{z} = & -\frac{\ddot{x}v_xv_z}{c^2} - \frac{\ddot{y}v_yv_z}{c^2} - \frac{\ddot{z}v_z^2}{c^2} - \frac{\ddot{z}v^2}{2c^2} - \\ & -\frac{3\ddot{z}GM}{c^2r} - \frac{3\ddot{z}GQ_2}{2c^2r^3} \left[-1 + 3 \left(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}} \right)^2 \right] + \frac{\partial\mathcal{L}_{\text{1PN}}}{\partial z}.\end{aligned}\quad (12)$$

All the other terms in the right-hand-sides of eq. (10)-eq. (12) which do not contain explicitly $\ddot{x}, \ddot{y}, \ddot{z}$ are included in

$$\frac{\partial\mathcal{L}_{\text{1PN}}}{\partial\mathbf{r}} = \nabla U - \frac{U\nabla U}{c^2} + \frac{3v^2\nabla U}{2c^2}.\quad (13)$$

They all contribute to the full 1PN acceleration of the test particle, including also terms quadratic in Q_2 which will be neglected in the following.

In order to have the correct expressions for the acceleration to the 1PN level, the second derivatives $\ddot{x}, \ddot{y}, \ddot{z}$ in the right-hand-sides of eq. (10)-eq. (12) must be replaced with their Newtonian expressions, as pointed out by Brumberg [2].

Thus, it is easy to show that the well-known 1PN point particle acceler-

ation

$$A_x^{(1\text{PN}0)} = \frac{GM \{4GMx + r [4v_x(v_x x + v_y y + v_z z) - v^2 x]\}}{c^2 r^4}, \quad (14)$$

$$A_y^{(1\text{PN}0)} = \frac{GM \{4GM y + r [4v_y(v_x x + v_y y + v_z z) - v^2 y]\}}{c^2 r^4}, \quad (15)$$

$$A_z^{(1\text{PN}0)} = \frac{GM \{4GM z + r [4v_z(v_x x + v_y y + v_z z) - v^2 z]\}}{c^2 r^4}. \quad (16)$$

causing the Einstein perihelion precession of Mercury is obtained just by summing up the 1PN terms of zero order in Q_2 in eq. (10)-eq. (12) containing $\ddot{x}, \ddot{y}, \ddot{z}$, computed with the Newtonian monopole acceleration

$$A_x^{(\text{N}0)} = -\frac{GMx}{r^3}, \quad (17)$$

$$A_y^{(\text{N}0)} = -\frac{GMy}{r^3}, \quad (18)$$

$$A_z^{(\text{N}0)} = -\frac{GMz}{r^3}, \quad (19)$$

to the other 1PN terms of zero order in Q_2 in eq. (13).

Some terms of order $\mathcal{O}(Q_2 c^{-2})$ arise by inserting eq. (17)-eq. (19) in the 1PN non-point particle terms in eq. (10)-eq. (12) containing $\ddot{x}, \ddot{y}, \ddot{z}$, and the Newtonian quadrupolar acceleration

$$\begin{aligned} A_x^{(\text{N}Q_2)} = & \frac{3GQ_2}{2r^7} \left\{ \left(1 - 3\hat{k}_x^2\right) x^3 - 8\hat{k}_x(\hat{k}_y y + \hat{k}_z z)x^2 + \right. \\ & + \left[\left(2\hat{k}_x^2 - 5\hat{k}_y^2 + 1\right) y^2 - 10\hat{k}_y \hat{k}_z z y + \left(2\hat{k}_x^2 - 5\hat{k}_z^2 + 1\right) z^2 \right] x + \\ & \left. + 2\hat{k}_x(\hat{k}_y y + \hat{k}_z z)(y^2 + z^2) \right\}, \end{aligned} \quad (20)$$

$$A_y^{(\text{N}Q_2)} = \frac{3GQ_2}{2r^7} \left\{ \left(1 - 3\hat{k}_y^2\right) y^3 - 8\hat{k}_y \hat{k}_z z y^2 - 5\hat{k}_x^2 x^2 y + \right.$$

$$\begin{aligned}
& + \left(2\hat{k}_y^2 - 5\hat{k}_z^2 + 1 \right) z^2 y + 2\hat{k}_y \hat{k}_z z^3 + x^2 \left(2y\hat{k}_y^2 + 2\hat{k}_z z \hat{k}_y + y \right) + \\
& + 2\hat{k}_x x \left[\hat{k}_y (x^2 - 4y^2 + z^2) - 5\hat{k}_z y z \right] \Big\}, \tag{21}
\end{aligned}$$

$$\begin{aligned}
A_z^{(\text{N}Q_2)} &= \frac{3GQ_2}{2r^7} \left\{ \left(1 - 3\hat{k}_z^2 \right) z^3 - 8\hat{k}_z (\hat{k}_x x + \hat{k}_y y) z^2 + \right. \\
& + \left[\left(-5\hat{k}_x^2 + 2\hat{k}_z^2 + 1 \right) x^2 - 10\hat{k}_x \hat{k}_y y x + \left(-5\hat{k}_y^2 + 2\hat{k}_z^2 + 1 \right) y^2 \right] z + \\
& \left. + 2\hat{k}_z (\hat{k}_x x + \hat{k}_y y) (x^2 + y^2) \right\} \tag{22}
\end{aligned}$$

in all the remaining 1PN terms of zero order in Q_2 in eq. (10)-eq. (12) before $\partial \mathcal{L}_{\text{PN}} / \partial \mathbf{r}$. It results an additional acceleration of order $\mathcal{O}(Q_2 c^{-2})$ with respect to that included in $\partial \mathcal{L}_{\text{PN}} / \partial \mathbf{r}$, not displayed here. Thus, the total 1PN quadrupole acceleration $\mathbf{A}^{(\text{1PN}Q_2)}$ is the sum of all of such contributions. Its components are

$$\begin{aligned}
A_x^{(\text{1PN}Q_2)} &= \frac{GQ_2}{2c^2 r^8} \left(8GM \left(\left(6\hat{k}_x^2 - 2 \right) x^3 + 15\hat{k}_x (\hat{k}_y y + \hat{k}_z z) x^2 + \right. \right. \\
& + \left. \left(\left(-3\hat{k}_x^2 + 9\hat{k}_y^2 - 2 \right) y^2 + 18\hat{k}_y \hat{k}_z z y + \left(-3\hat{k}_x^2 + 9\hat{k}_z^2 - 2 \right) z^2 \right) x - \right. \\
& - 3\hat{k}_x (\hat{k}_y y + \hat{k}_z z) (y^2 + z^2) \Big) + 3r \left(3v_x^2 \left(\left(3\hat{k}_x^2 - 1 \right) x^3 + \right. \right. \\
& + 8\hat{k}_x (\hat{k}_y y + \hat{k}_z z) x^2 + \left(\left(-2\hat{k}_x^2 + 5\hat{k}_y^2 - 1 \right) y^2 + 10\hat{k}_y \hat{k}_z z y + \right. \\
& + \left. \left(-2\hat{k}_x^2 + 5\hat{k}_z^2 - 1 \right) z^2 \right) x - 2\hat{k}_x (\hat{k}_y y + \hat{k}_z z) (y^2 + z^2) \Big) - \\
& - (v_y^2 + v_z^2) \left(\left(3\hat{k}_x^2 - 1 \right) x^3 + 8\hat{k}_x (\hat{k}_y y + \hat{k}_z z) x^2 + \right. \\
& + \left. \left(\left(-2\hat{k}_x^2 + 5\hat{k}_y^2 - 1 \right) y^2 + 10\hat{k}_y \hat{k}_z z y + \left(-2\hat{k}_x^2 + 5\hat{k}_z^2 - 1 \right) z^2 \right) x - \right.
\end{aligned}$$

$$\begin{aligned}
& -2\hat{k}_x(\hat{k}_y y + \hat{k}_z z)(y^2 + z^2) + 4v_x \left(5\hat{k}_x^2(v_y y + v_z z)x^2 - \right. \\
& -2\hat{k}_x \left(\hat{k}_z(v_z(x^2 + y^2 - 4z^2) - 5v_y y z) + \hat{k}_y(v_y(x^2 - 4y^2 + z^2) - \right. \\
& -5v_z y z)) x - v_y \left((1 - 3\hat{k}_y^2)y^3 - 8\hat{k}_y \hat{k}_z z y^2 + (2\hat{k}_y^2 - 5\hat{k}_z^2 + 1)z^2 y + \right. \\
& + 2\hat{k}_y \hat{k}_z z^3 + x^2(2y\hat{k}_y^2 + 2\hat{k}_z z \hat{k}_y + y)) - v_z \left((1 - 3\hat{k}_z^2)z^3 - \right. \\
& \left. \left. -5\hat{k}_y^2 y^2 z + (2\hat{k}_z^2 + 1)(x^2 + y^2)z + 2\hat{k}_y \hat{k}_z y(x^2 + y^2 - 4z^2) \right) \right) \Big), \\
& \tag{23}
\end{aligned}$$

$$\begin{aligned}
A_y^{(1\text{PN}Q_2)} = & -\frac{GQ_2}{2c^2 r^8} \left(12v_x v_y x r^3 + 3 \left(-2 \left(\hat{k}_y \hat{k}_z (v^2 - 4v_y^2) + \right. \right. \right. \\
& + 2 \left(3\hat{k}_z^2 - 1 \right) v_y v_z \Big) z^3 - \left(20v_x v_y x \hat{k}_z^2 + \left((2\hat{k}_y^2 - 5\hat{k}_z^2 + 1)(v^2 - \right. \right. \\
& - 4v_y^2) + 32\hat{k}_y \hat{k}_z v_y v_z \Big) y \Big) z^2 + 2 \left(-10\hat{k}_y^2 v_y v_z y^2 + \right. \\
& + 2 \left(2\hat{k}_z^2 + 1 \right) v_y v_z (x^2 + y^2) + \hat{k}_y \hat{k}_z (4v_y(v_y x^2 - 5v_x y x - 4v_y y^2) - \\
& - v^2(x^2 - 4y^2)) \Big) z + y \left(-20v_x v_y x y \hat{k}_y^2 + \left(8\hat{k}_y \hat{k}_z v_y v_z - \right. \right. \\
& - \left(2\hat{k}_y^2 + 1 \right) (v^2 - 4v_y^2) \Big) x^2 + \left((3\hat{k}_y^2 - 1)(v^2 - 4v_y^2) + \right. \\
& \left. \left. + 8\hat{k}_y \hat{k}_z v_y v_z \right) y^2 \right) + \hat{k}_x^2 x (5x(v^2 y - 4v_y(v_y y + v_z z)) +
\end{aligned}$$

$$\begin{aligned}
& +4v_x v_y (2(y^2 + z^2) - 3x^2)) + 2\hat{k}_x (\hat{k}_z (5xyzv^2 - \\
& -20v_y^2 xyz + 4v_y (v_z x (x^2 + y^2 - 4z^2) + v_x z (-4x^2 + y^2 + z^2))) - \\
& -\hat{k}_y (v^2 x (x^2 - 4y^2 + z^2) - 4v_y (v_y x (x^2 - 4y^2 + z^2) + \\
& +y (v_x (-4x^2 + y^2 + z^2) - 5v_z xz)))))) r + \\
& +8GM \left(2 \left(1 - 3\hat{k}_y^2 \right) y^3 - 9\hat{k}_x^2 x^2 y + \left(3\hat{k}_y^2 + 2 \right) x^2 y + \right. \\
& + \left(3\hat{k}_y^2 - 9\hat{k}_z^2 + 2 \right) z^2 y + 3\hat{k}_y \hat{k}_z z^3 + 3\hat{k}_y \hat{k}_z (x^2 - 5y^2) z + \\
& \left. 3\hat{k}_x x (\hat{k}_y (x^2 - 5y^2 + z^2) - 6\hat{k}_z yz) \right), \tag{24}
\end{aligned}$$

$$\begin{aligned}
A_z^{(1\text{PN}Q_2)} = & -\frac{GQ_2}{2c^2 r^8} \left(8GM \left(-2 \left(3\hat{k}_z^2 - 1 \right) z^3 - 15\hat{k}_z (\hat{k}_x x + \hat{k}_y y) z^2 + \right. \right. \\
& + \left(\left(-9\hat{k}_x^2 + 3\hat{k}_z^2 + 2 \right) x^2 - 18\hat{k}_x \hat{k}_y yx + \left(-9\hat{k}_y^2 + 3\hat{k}_z^2 + 2 \right) y^2 \right) z + \\
& + 3\hat{k}_z (\hat{k}_x x + \hat{k}_y y) (x^2 + y^2) \Big) + 3r \left(-2 \left(6v_y v_z \hat{k}_y^2 + \hat{k}_z (v^2 - 4v_z^2) \hat{k}_y - \right. \right. \\
& - 2v_y v_z) y^3 + 2 \left(2v_y v_z + \hat{k}_y (-\hat{k}_z v^2 + 4\hat{k}_z v_z^2 + 4\hat{k}_y v_y v_z) \right) x^2 y + \\
& + \left(\left(3\hat{k}_z^2 - 1 \right) v^2 + 4v_z \left(-3v_z \hat{k}_z^2 + 2\hat{k}_y v_y \hat{k}_z + v_z \right) \right) z^3 - \\
& - 4 \left(\left(5\hat{k}_z^2 - 1 \right) v_x v_z x - \left(2\hat{k}_y \hat{k}_z v^2 - 8\hat{k}_y \hat{k}_z v_z^2 + \right. \right. \\
& + \left. \left. \left(2\hat{k}_y^2 - 5\hat{k}_z^2 + 1 \right) v_y v_z \right) y \right) z^2 + 4v_x v_z x \left(x^2 + \left(1 - 5\hat{k}_y^2 \right) y^2 \right) -
\end{aligned}$$

$$\begin{aligned}
& - \left(\left(\left(2\hat{k}_z^2 + 1 \right) x^2 + \left(-5\hat{k}_y^2 + 2\hat{k}_z^2 + 1 \right) y^2 \right) v^2 + 4v_z \left(5\hat{k}_y^2 v_z y^2 - \right. \right. \\
& - \left. \left(2\hat{k}_z^2 + 1 \right) v_z (x^2 + y^2) + 2\hat{k}_y \hat{k}_z (-v_y x^2 + 5v_x yx + 4v_y y^2) \right) \right) z + \\
& + \hat{k}_x^2 x \left(5x (v^2 z - 4v_z (v_y y + v_z z)) + 4v_x v_z (2(y^2 + z^2) - 3x^2) \right) + \\
& + 2\hat{k}_x \left(\hat{k}_y (4v_y v_z x (x^2 - 4y^2 + z^2) + y (5(v^2 - 4v_z^2) xz + \right. \\
& + 4v_x v_z (-4x^2 + y^2 + z^2))) + \hat{k}_z (4v_z (v_z x (x^2 + y^2 - 4z^2) + \\
& + z (v_x (-4x^2 + y^2 + z^2) - 5v_y xy)) - v^2 x (x^2 + y^2 - 4z^2))) \right). \tag{25}
\end{aligned}$$

It can be shown that eq. (23)-eq. (25) are in agreement with [6]. They also agree with¹ the acceleration \mathbf{F}_4 at pag. 112 of [2], whose calculation is, however, restricted to the case $\hat{\mathbf{k}} = \hat{\mathbf{z}}$.

In the literature [1–5, 9], perturbative calculations of the direct orbital perturbations of order $\mathcal{O}(Q_2 c^{-2})$ exist in the particular case $\hat{\mathbf{k}} = \hat{\mathbf{z}}$. Since in many astronomical and astrophysical scenarios of potential interest the actual orientation in space of $\hat{\mathbf{k}}$ is either unknown or partially known, here we will look at the consequences of eq. (23)-eq. (25) without restricting to any specific direction of $\hat{\mathbf{k}}$.

3 The direct post-Newtonian quadrupole orbital effects of order $\mathcal{O}(Q_2 c^{-2})$ for a generic orientation of $\hat{\mathbf{k}}$ in space

We now consider the direct orbital effects of eq. (23)-eq. (25) as if they were the only perturbing acceleration \mathbf{A} of the Newtonian monopole, i.e. we will neglect both eq. (20)-eq. (22) and eq. (14)-eq. (16). The resulting long-term orbital effects can be analytically worked out with the Gauss equations

¹The quadrupole Q in [2] corresponds to GQ_2 here.

for the variations of the osculating Keplerian orbital elements [2]. They are

$$\frac{da}{dt} = \frac{2}{n_b \sqrt{1-e^2}} \left(e A_R \sin f + \frac{p}{r} A_T \right), \quad (26)$$

$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{n_b a} \left\{ A_R \sin f + A_T \left[\cos f + \frac{1}{e} \left(1 - \frac{r}{a} \right) \right] \right\}, \quad (27)$$

$$\frac{dI}{dt} = \frac{r \cos u}{n_b a^2 \sqrt{1-e^2}} A_N, \quad (28)$$

$$\frac{d\Omega}{dt} = \frac{r \sin u}{n_b a^2 \sin I \sqrt{1-e^2}} A_N, \quad (29)$$

$$\frac{d\omega}{dt} = -\cos I \frac{d\Omega}{dt} + \frac{\sqrt{1-e^2}}{n_b a e} \left[-A_R \cos f + A_T \left(1 + \frac{r}{p} \right) \sin f \right]. \quad (30)$$

In eq. (26)-eq. (30), f is the true anomaly, a is the semimajor axis, $n_b \doteq \sqrt{GMa^{-3}}$ is the Keplerian mean motion, e is the eccentricity, $p \doteq a(1-e^2)$ is the semilatus rectum, I is the inclination, Ω is the longitude of the ascending node, ω is the argument of the pericenter, $u \doteq \omega + f$ is the argument of the latitude, A_R, A_T, A_N are the radial, transverse and normal components of the disturbing acceleration \mathbf{A} , respectively. It is intended that the right-hand-sides of eq. (26)-eq. (30) have to be evaluated onto the unperturbed Keplerian ellipse

$$r = \frac{p}{1 + e \cos f}. \quad (31)$$

In principle, the time average of eq. (26)-eq. (30) should be done by adopting [2, 6]

$$\frac{df}{dt} = \frac{n_b a^2}{r^2} \sqrt{1-e^2} - \left(\frac{d\omega}{dt} + \cos I \frac{d\Omega}{dt} \right), \quad (32)$$

where f is the true anomaly, and the derivatives of Ω and ω are to be taken from eq. (29)-eq. (30) themselves. It is so because f is counted from the pericenter location, which, in general, changes because of the variations of Ω and ω [6]. If just a first-order accuracy with respect to the disturbing acceleration A is required, the averages of eq. (26)-eq. (30) over one orbit revolution can be obtained with [2]

$$\frac{df}{dt} = n_b \left(\frac{a}{r} \right)^2 \sqrt{1-e^2} \quad (33)$$

by integrating over f between 0 and 2π .

After a lengthy calculation, the direct averaged orbital rates of change of order $\mathcal{O}(Q_2 c^{-2})$ turn out to be

$$\begin{aligned}
\left\langle \frac{da}{dt} \right\rangle = & \frac{9GQ_2 n_b e^2 (6 + e^2)}{32c^2 a^2 (1 - e^2)^4} \left\{ -8\hat{k}_z \sin I \cos 2\omega \left(\hat{k}_x \cos \Omega + \hat{k}_y \sin \Omega \right) + \right. \\
& + 4 \cos I \cos 2\omega \left[-2\hat{k}_x \hat{k}_y \cos 2\Omega + \left(\hat{k}_x^2 - \hat{k}_y^2 \right) \sin 2\Omega \right] + \\
& + \sin 2\omega \left[\left(\hat{k}_x^2 - \hat{k}_y^2 \right) (3 + \cos 2I) \cos 2\Omega + 2 \left(1 - 3\hat{k}_z^2 \right) \sin^2 I - \right. \\
& - 4\hat{k}_z \sin 2I \left(\hat{k}_y \cos \Omega - \hat{k}_x \sin \Omega \right) + \\
& \left. \left. + 2\hat{k}_x \hat{k}_y (3 + \cos 2I) \sin 2\Omega \right] \right\}, \tag{34}
\end{aligned}$$

$$\begin{aligned}
\left\langle \frac{de}{dt} \right\rangle = & -\frac{21GQ_2 n_b e (2 + e^2)}{64c^2 a^3 (1 - e^2)^3} \left\{ 8\hat{k}_z \sin I \cos 2\omega \left(\hat{k}_x \cos \Omega + \hat{k}_y \sin \Omega \right) + \right. \\
& + 4 \cos I \cos 2\omega \left[2\hat{k}_x \hat{k}_y \cos 2\Omega - \left(\hat{k}_x^2 - \hat{k}_y^2 \right) \sin 2\Omega \right] - \\
& - \sin 2\omega \left[\left(\hat{k}_x^2 - \hat{k}_y^2 \right) (3 + \cos 2I) \cos 2\Omega + 2 \left(1 - 3\hat{k}_z^2 \right) \sin^2 I - \right. \\
& - 4\hat{k}_z \sin 2I \left(\hat{k}_y \cos \Omega - \hat{k}_x \sin \Omega \right) + \\
& \left. \left. + 2\hat{k}_x \hat{k}_y (3 + \cos 2I) \sin 2\Omega \right] \right\}, \tag{35}
\end{aligned}$$

$$\left\langle \frac{dI}{dt} \right\rangle = -\frac{3GQ_2 n_b}{4c^2 a^3 (1 - e^2)^3} \left[\hat{k}_z \cos I + \sin I \left(\hat{k}_x \sin \Omega - \hat{k}_y \cos \Omega \right) \right]$$

$$\begin{aligned}
& \left[e^2 \hat{k}_z \sin I \sin 2\omega + e^2 \cos I \sin 2\omega \left(\hat{k}_y \cos \Omega - \hat{k}_x \sin \Omega \right) + \right. \\
& \left. + (6 + e^2 \cos 2\omega) \left(\hat{k}_x \cos \Omega + \hat{k}_y \sin \Omega \right) \right], \tag{36}
\end{aligned}$$

$$\begin{aligned}
\left\langle \frac{d\Omega}{dt} \right\rangle &= \frac{3GQ_2 n_b \csc I}{4c^2 a^3 (1 - e^2)^3} \left[\hat{k}_z \cos I + \sin I \left(\hat{k}_x \sin \Omega - \hat{k}_y \cos \Omega \right) \right] \\
& \left\{ (-6 + e^2 \cos 2\omega) \left[\hat{k}_z \sin I + \cos I \left(\hat{k}_y \cos \Omega - \hat{k}_x \sin \Omega \right) \right] - \right. \\
& \left. - e^2 \sin 2\omega \left(\hat{k}_x \cos \Omega + \hat{k}_y \sin \Omega \right) \right\}, \tag{37}
\end{aligned}$$

$$\begin{aligned}
\left\langle \frac{d\omega}{dt} \right\rangle &= -\frac{3GQ_2 n_b}{64c^2 a^3 (1 - e^2)^3} \left\{ (8 - 3e^2) \left(1 - 3\hat{k}_z^2 \right) + \right. \\
& + 12 (3e^2 - 8) \hat{k}_z \sin 2I \left(\hat{k}_x \sin \Omega - \hat{k}_y \cos \Omega \right) - \\
& - 112 \hat{k}_z \sin I \sin 2\omega (\hat{k}_x \cos \Omega + \hat{k}_y \sin \Omega) - \\
& - 14 \cos 2\omega \left[1 - 3\hat{k}_z^2 + 4\hat{k}_z \sin 2I (\hat{k}_x \sin \Omega - \hat{k}_y \cos \Omega) \right] + \\
& + 16 \cot I [\hat{k}_z \cos I + \sin I (\hat{k}_x \sin \Omega - \hat{k}_y \cos \Omega)] \\
& \left[-e^2 \sin 2\omega \left(\hat{k}_x \cos \Omega + \hat{k}_y \sin \Omega \right) + \hat{k}_z (e^2 \cos 2\omega - 6) \sin I + \right. \\
& + \cos I (e^2 \cos 2\omega - 6) \left(\hat{k}_y \cos \Omega - \hat{k}_x \sin \Omega \right) \left. \right] - \\
& - 3 (3e^2 + 14 \cos 2\omega - 8) \left[\left(\hat{k}_x^2 - \hat{k}_y^2 \right) \cos 2\Omega + 2\hat{k}_x \hat{k}_y \sin 2\Omega \right] +
\end{aligned}$$

$$\begin{aligned}
& + \cos 2I (9e^2 - 14 \cos 2\omega - 24) \left[2\hat{k}_x \hat{k}_y \sin 2\Omega - 1 + 3\hat{k}_z^2 + \right. \\
& + \left(\hat{k}_x^2 - \hat{k}_y^2 \right) \cos 2\Omega \left. \right] + 56 \cos I \sin 2\omega \left[\left(\hat{k}_x^2 - \hat{k}_y^2 \right) \sin 2\Omega - \right. \\
& \left. - 2\hat{k}_x \hat{k}_y \cos 2\Omega \right] \left. \right\}. \tag{38}
\end{aligned}$$

Note that eq. (34)-eq. (38) have a general validity since they hold for an arbitrary orientation of $\hat{\mathbf{k}}$ in space. In the particular case $\hat{\mathbf{k}} = \hat{\mathbf{z}}$, eq. (34)-eq. (38) reduce to

$$\frac{da}{dt} = -\frac{9GQ_2n_b e^2 (6 + e^2) \sin^2 I \sin 2\omega}{8c^2 a^2 (1 - e^2)^4}, \tag{39}$$

$$\frac{de}{dt} = -\frac{21GQ_2n_b e (2 + e^2) \sin^2 I \sin 2\omega}{16c^2 a^3 (1 - e^2)^3}, \tag{40}$$

$$\frac{dI}{dt} = -\frac{3GQ_2n_b e^2 \sin 2I \sin 2\omega}{8c^2 a^3 (1 - e^2)^3}, \tag{41}$$

$$\frac{d\Omega}{dt} = -\frac{3GQ_2n_b \cos I (6 - e^2 \cos 2\omega)}{4c^2 a^3 (1 - e^2)^3}, \tag{42}$$

$$\begin{aligned}
\frac{d\omega}{dt} = & \frac{3GQ_2n_b}{32c^2 a^3 (1 - e^2)^3} \{ 32 - 3e^2 - 2(7 + 2e^2) \cos 2\omega + \\
& + \cos 2I [48 - 9e^2 + 2(7 - 2e^2) \cos 2\omega] \}. \tag{43}
\end{aligned}$$

The special formulas of eq. (39)-eq. (43) were previously obtained in² [1–3] with the Gauss equations.

3.1 Integrating the equations of motion: a numerical test

To independently check eq. (34)-eq. (38), we will treat the direct 1PN effects of order $\mathcal{O}(Q_2 c^{-2})$ numerically by integrating the equations of motion of a

² Note that the expression in the right-hand-side of eq. (1) in [9] actually corresponds to $d \ln a / dt$, not to da / dt , as erroneously written in [9].

fictional star orbiting, say, the supermassive black hole located at the center of the Galaxy in Sgr A* with and without eq. (23)-eq. (25), in addition to eq. (17)-eq. (19). Both the integrations share the same initial conditions. Figure 1-Figure 2 display the resulting temporal evolutions of the standard Keplerian orbital elements for different values of the right ascension (RA) α and declination (DEC) δ of $\hat{\mathbf{k}}$ parameterized as

$$\hat{k}_x = \cos \alpha \cos \delta, \quad (44)$$

$$\hat{k}_y = \sin \alpha \cos \delta, \quad (45)$$

$$\hat{k}_z = \sin \delta. \quad (46)$$

During the whole integrations, the condition $|A^{(\text{IPN}Q_2)}/A^{(\text{N}0)}| \ll 1$ was fulfilled, so that a comparison with the perturbative analytical calculations of eq. (34)-eq. (38) can be meaningfully made. From Figure 1-Figure 2, it can be noticed that the evolution of the orbital elements is linear in time. Linear fits to the numerically integrated signals yield slopes as large as those predicted by the analytical formulas of eq. (34)-eq. (38), thus supporting their validity.

4 The indirect, mixed post-Newtonian quadrupole orbital perturbations of order $\mathcal{O}(Q_2 c^{-2})$

As remarked in [1, 4, 6], orbital perturbations of order $\mathcal{O}(Q_2 c^{-2})$ arise, in principle, even from just eq. (20)-eq. (22) and eq. (14)-eq. (16), leaving eq. (23)-eq. (25) aside. Such indirect, mixed effects have a twofold origin; see also the discussion in [6].

First, by taking the sum of both eq. (20)-eq. (22) and eq. (14)-eq. (16) in the disturbing acceleration \mathbf{A} to be inserted in eq. (26)-eq. (30), the use of eq. (32) in calculating their long-term rates of change induces mixed terms³ proportional to $Q_2 c^{-2}$ which may have, in general, the same order of magnitude as the direct ones worked out in Section 3. Notice that, in performing such calculations, the Keplerian orbital elements are to be kept fixed in the integration with respect to f .

Indirect effects of order $\mathcal{O}(Q_2 c^{-2})$ generally arise also by adopting eq. (33), provided that the f -dependent short-term shifts of the Keplerian or-

³ Also higher order terms in Q_2 and c^{-2} arise; they will be neglected.

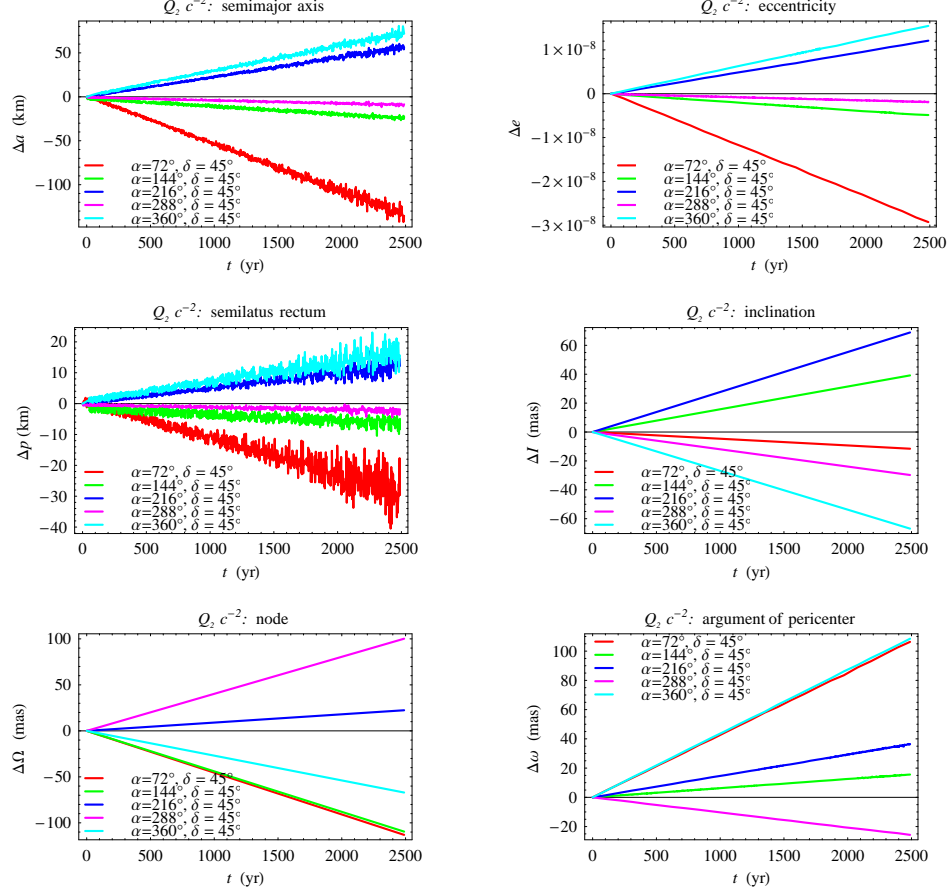


Figure 1: Temporal evolution of the standard Keplerian orbital elements, where $p = a(1 - e^2)$ is the semilatus rectum, of a hypothetical test particle orbiting an oblate body with mass $M = 4.1 \times 10^6 M_\odot$, Schwarzschild radius $R_S = 0.08$ au, quadrupole moment $Q_2 = 1815 M_\odot \text{ au}^2$ for different spatial orientations of its spin axis $\hat{\mathbf{k}}$ parameterized in terms of the right ascension (RA) α and declination (DEC) δ . DEC was kept fixed to $\delta = 45^\circ$. The Keplerian orbital period of the fictional star considered is $P_b = 0.7$ yr. The equations of motion were numerically integrated in Cartesian coordinates with and without $\mathbf{A}^{(1\text{PN}Q_2)}$ starting from the same initial conditions corresponding to $r_{\min} = 1500 R_S$, $e_0 = 0.09$, $I_0 = 25^\circ$, $\Omega_0 = 0^\circ$, $\omega_0 = 45^\circ$, $f_0 = 0^\circ$. In order to single out just the direct effects of order $\mathcal{O}(Q_2 c^{-2})$, neither $\mathbf{A}^{(NQ_2)}$ nor $\mathbf{A}^{(1\text{PN}0)}$ were included in the numerical integrations. During the whole integrations, the perturbative condition $|A^{(1\text{PN}Q_2)}/A^{(N0)}| \ll 1$ was fulfilled.

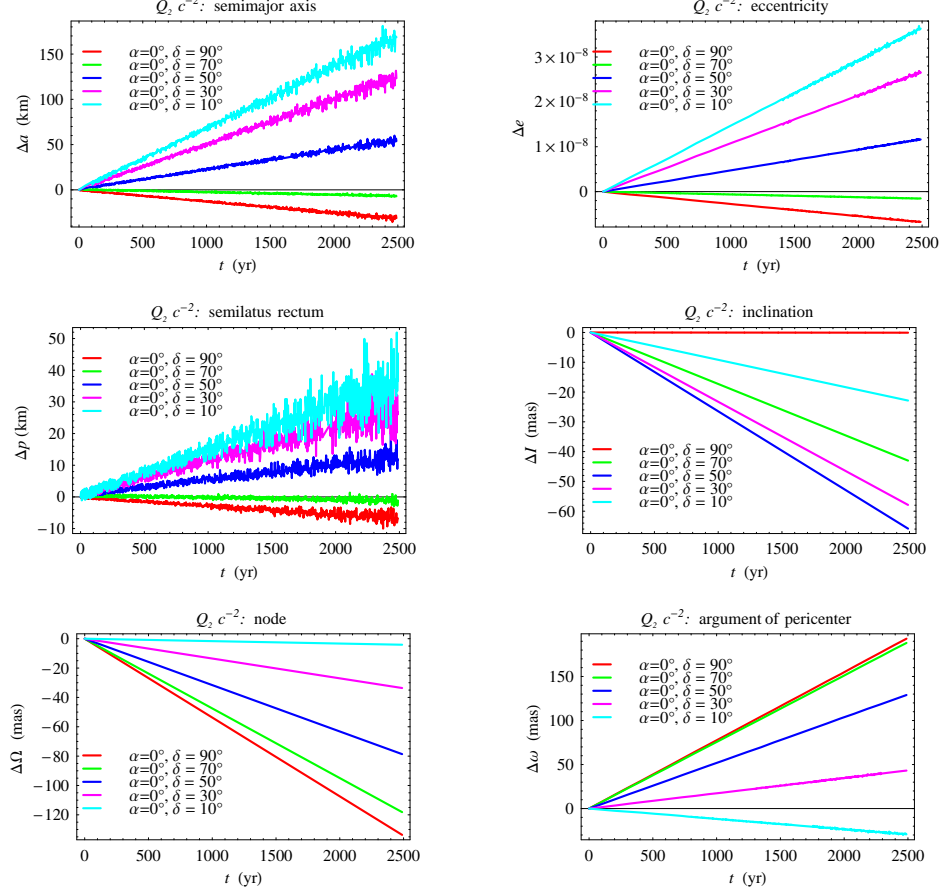


Figure 2: Temporal evolution of the standard Keplerian orbital elements, where $p = a(1 - e^2)$ is the semilatus rectum, of a hypothetical test particle orbiting an oblate body with mass $M = 4.1 \times 10^6 M_\odot$, Schwarzschild radius $R_S = 0.08$ au, quadrupole moment $Q_2 = 1815 M_\odot \text{ au}^2$ for different spatial orientations of its spin axis $\hat{\mathbf{k}}$ parameterized in terms of the right ascension (RA) α and declination (DEC) δ . RA was kept fixed to $\alpha = 0^\circ$. The Keplerian orbital period of the fictional star considered is $P_b = 0.7$ yr. The equations of motion were numerically integrated in Cartesian coordinates with and without $\mathbf{A}^{(1\text{PN}Q_2)}$ starting from the same initial conditions corresponding to $r_{\min} = 1500 R_S$, $e_0 = 0.09$, $I_0 = 25^\circ$, $\Omega_0 = 0^\circ$, $\omega_0 = 45^\circ$, $f_0 = 0^\circ$. In order to single out just the direct effects of order $\mathcal{O}(Q_2 c^{-2})$, neither $\mathbf{A}^{(NQ_2)}$ nor $\mathbf{A}^{(1\text{PN}0)}$ were included in the numerical integrations. During the whole integrations, the perturbative condition $|A^{(1\text{PN}Q_2)}/A^{(N0)}| \ll 1$ was fulfilled.

bit elements due to eq. (20)-eq. (22) are included in the calculation of the rates of change due to eq. (14)-eq. (16), and vice-versa.

Actually, we will not treat all such kinds of mixed perturbations further because they are essentially undetectable in practical data reductions of concrete astronomical systems. Indeed, let us think about, e.g., an artificial satellite orbiting an oblate planet. Data analysts routinely model all Newtonian multipolar accelerations and the 1PN point particle acceleration to the best of our current knowledge of the parameters entering them which, of course, is necessarily imperfect. Thus, the actual satellite residuals of, say, ranges, range-rates, Keplerian orbital elements, etc. would not show the indirect, mixed Q_2c^{-2} effects in full. They could only contain negligible signatures due to the mismodeling in the quadrupole moment and in the 1PN point particle acceleration. Instead, at least in principle, they should fully display the direct Q_2c^{-2} effects (unless they have been somewhat removed in the data reduction procedure), which are usually not included in the dynamical models fit to the observations. Otherwise, one should not model either eq. (20)-eq. (22) and eq. (14)-eq. (16) at all, and subtract their theoretically computed signals from the resulting huge residuals. It does not seem viable because of the unavoidable uncertainties. Even from the point of view of a covariance analysis, while it would be possible, in principle, to solve for a dedicated scaling parameter accounting for $\mathbf{A}^{(1\text{PN}Q_2)}$, this could not be done for the indirect effects.

5 Summary and conclusions

We analytically worked out the direct post-Newtonian long-term rates of change of the motion of a test particle orbiting an oblate primary by assuming an arbitrary orientation in space of its spin axis $\hat{\mathbf{k}}$. Our results, obtained with the Gauss equations for the variation of the orbital elements, are in agreement with the existing formulas valid for the special case in which $\hat{\mathbf{k}}$ is aligned with the reference z axis. We successfully checked our analytical calculation by numerically integrating the equations of motion of a fictitious point particle acted upon by the post-Newtonian acceleration arising from the oblateness of its primary. Since no a priori assumptions on both $\hat{\mathbf{k}}$ and the orbital configuration were made, our formulas have a general validity and can be extended to a variety of astronomical and astrophysical systems.

We discussed the indirect, mixed effects arising from the interplay between the Newtonian quadrupole and the post-Newtonian Schwarzschild-like

acceleration. We remarked that they are not measurable in actual data reductions. Indeed, the residuals, obtained by routinely modeling standard Newtonian and post-Newtonian dynamics would contain just negligible signatures due to the mismodeling in them. On the contrary, the effects induced directly by the post-Newtonian quadrupole acceleration, usually not modeled, would fully impact the residuals, at least in principle.

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